

Some Fractal Properties of the Percolating Backbone in Two Dimensions

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A new algorithm is presented, based on elements of artificial intelligence theory, to determine the fractal properties of the backbone of the incipient infinite cluster. It is found that the fractal dimensionality of the backbone is $d_f^{BB} = 1.61 \pm 0.01$, the chemical dimensionality is $d_c = 1.40 \pm 0.01$, and the fractal dimension of the minimum path $d_{\min} = 1.15 \pm 0.02$ for the two-dimensional triangular lattice.

KEY WORDS: Percolation; incipient infinite cluster; backbone; fractal properties; artificial intelligence.

1. INTRODUCTION

The fractal properties of the percolating cluster at the percolation threshold p_c have recently been the focus of careful scrutiny, particularly the evaluation of the critical exponents necessary to describe the backbone of this incipient infinite cluster (IIC). For a recent review of the fractal properties of the incipient infinite clusters see, e.g., Refs. 1 and 2 and for the critical properties of the backbone see, e.g., Refs. 3–6. The backbone may be defined as the union of all self-avoiding walks defined between extremities of the IIC at the percolation threshold; or an alternative and equally viable description is the set of current-carrying bonds of the percolating cluster when a potential difference is placed at the extremities and each bond represents a unit resistance (the so-called random resistor network). The non-current-carrying bonds are referred to as dangling ends.

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There is an equivalent description for site percolation, where the backbone now refers to the set of sites through which the current flows and where all nearest neighbor occupied sites are connected by bonds of unit resistance. Recent work⁽³⁻⁶⁾ has shown that an infinite hierarchy or spectrum of exponents is necessary to describe how the various moments of the voltage distribution of the bonds of the backbone scale at the percolation threshold, i.e., the moments of the voltage distribution scale independently and not, as one might intuitively expect from ordinary critical phenomena, with constant "gap" scaling.

The backbone is also of interest, since its geometrical properties have led to a more comprehensive understanding of the general structure of percolating clusters, e.g., the blobs, links, and nodes picture is primarily due to our understanding of the internal structure of the backbone. The fractal dimensionality of the backbone d_f^{BB} of the IIC has been determined numerically by several methods: large cell renormalization group,⁽⁷⁾ ϵ expansion,^(8,9) position-space renormalization group,⁽¹⁰⁾ and series expansion.⁽¹¹⁾ A range of values was reported for d_f^{BB} from 1.55 to 1.80. Recently Herrmann *et al.*⁽¹⁴⁾ introduced a new algorithm to determine the backbone of the percolating cluster for the two-dimensional square lattice and reported a value for d_f^{BB} of 1.60 ± 0.05 . In a subsequent letter devoted to the analysis of the blob distribution function, Herrmann and Stanley⁽¹⁵⁾ were able to obtain a more precise value for d_f^{BB} (1.62 ± 0.02), which was obtained from a precise determination of $\tilde{\tau}$ and the scaling relationship

$$\tilde{\tau} - 1 = d_t/d_f^{\text{BB}} \quad (1)$$

where $\tilde{\tau}$ describes how the number of blobs of size s scales with s for a given system size L and d_t describes how the number of blobs scales with L .

In this paper we report on an independent and new algorithm to determine the fractal dimensionality of the backbone. The algorithm described below is used to construct the backbone from clusters grown via the Leath^(16,17) algorithm. The clusters are grown on the triangular lattice and in addition to the fractal dimensionality we determine the chemical dimensionality d_t and the exponent describing how the radius of gyration of the backbone scales with chemical distance, $\tilde{\nu}$. We are also able to deduce $d_{\text{min}} = 1/\tilde{\nu}$,⁽¹⁸⁻²¹⁾ the exponent describing the minimum path between arbitrarily selected points on the backbone. For a comprehensive description of these exponents we refer the reader to Stanley.⁽²²⁾

The numerical results for the minimum path exponent reported for percolation varied from 1.18 to 1.10 (see Ref. 19 for references). Havlin and Nossal⁽¹⁸⁾ have conjectured that $d_{\text{min}} = 2 - 1/\nu - \beta/\nu = 1.1458$, which is in

good agreement with most numerical estimates. However, Grassberger⁽¹⁹⁾ has recently presented numerical results for the chemical dimensionality of the percolation cluster. This is related to d_{\min} via

$$\hat{d} = (2\nu - \beta)/\nu^{11}$$

where $d_{\min} = \nu^{11}/\nu$. Here ν is the correlation length exponent, β the order parameter exponent, and ν^{11} is the ratio $\nu/\tilde{\nu}$. The value obtained for \hat{d} leads to $d_{\min} = 1.132 \pm 0.003$ and thus excludes the conjecture of Ref. 18. However, the work of Edwards and Kerstein^(20,21) from the consideration of first passage velocity on the square lattice concluded that $d_{\min} = 1$, a value in complete disagreement with all previous work.

2. THE RECURSIVE ALGORITHM

The algorithm to determine the backbone is a recursive one, which is implemented in two stages. The first step involves the complete description of the cluster in terms of a treelike structure (see Fig. 1a and 1b) and thus determines a path from one endpoint (the root) P_s of the cluster to the other endpoint P_1 (Fig. 1c). This path may be considered as the "backbone" of the backbone. We have constructed a linear chain of sites P_i , which is a part of the backbone. The second step consists in finding all paths that leave the partially constructed backbone $\{P_i\}$ at one site and return to the backbone at some other site. These sites or self-avoiding walk segments are now added onto the growing backbone and the process continued until no further points are found. This is the recursive part of the algorithm, which was written in Pascal. A version was also completed in Modula-2. In order to implement the algorithm, it is convenient to construct a directed tree structure to define the cluster of occupied sites with the root of the tree as one of the endpoints of the backbone. Each site of the cluster is represented as a node of the tree and at each node information is stored to direct to neighboring nodes and also a pointer to its parent node. There is, therefore, a unique path from one of the endpoints (the root) to each site of the cluster. The tree structure allows us to traverse the cluster efficiently in either direction. This representation of the cluster by a tree structure defining a unique path to every site of the cluster enables the implementation of a heuristic search, common in artificial intelligence applications and expert systems.

The algorithm used to grow the tree and hence to determine the backbone is a modification of the A* algorithm,^(23,24) which guarantees the finding of a solution if one exists. Figure 1 displays the various stages of the backbone algorithm. Our programs consist in (1) growing a cluster on the triangular lattice to a certain chemical distance, (2) determining the sites

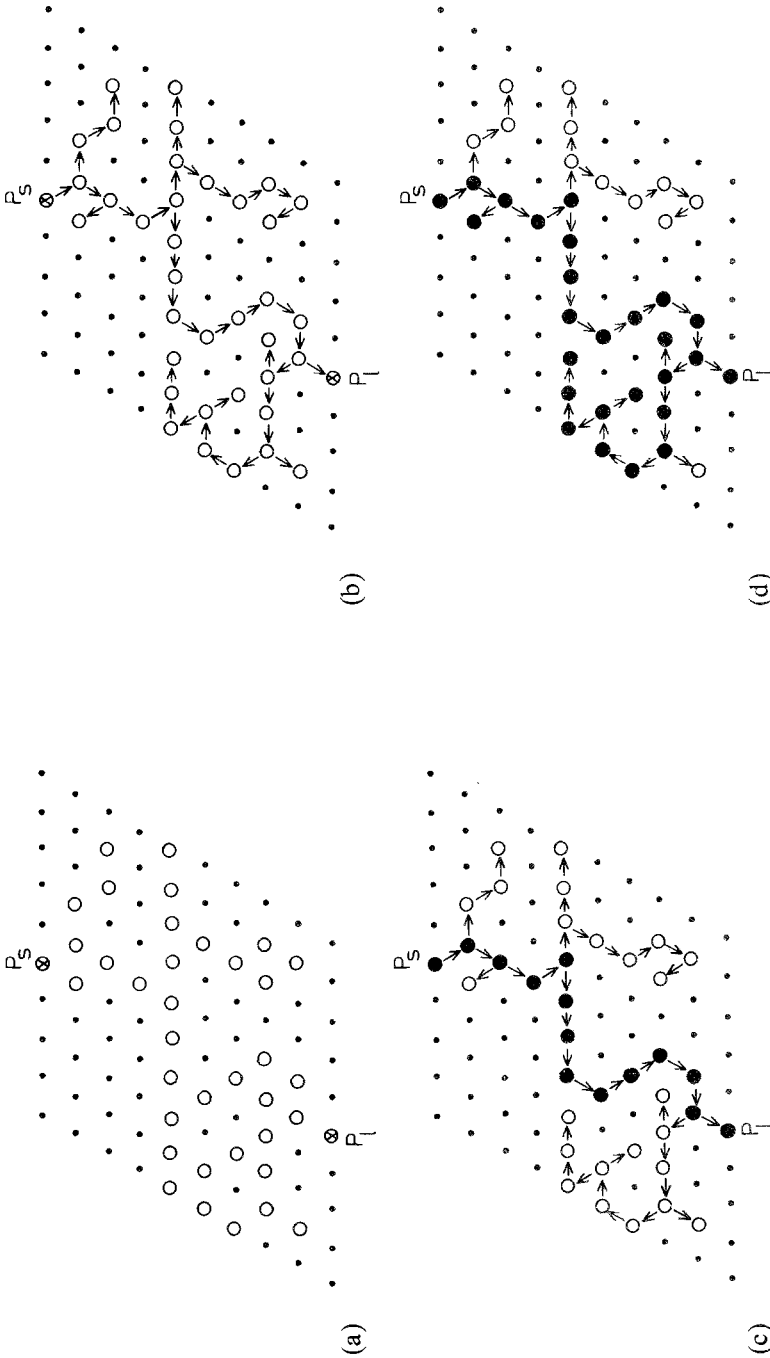


Fig. 1. The construction of the backbone. (a) Mark the cluster endpoints, (b) create the tree structure, (c) construct a "backbone" of the backbone, (d) for every \circ site that has a \bullet site as one of its neighbors (excluding its ancestor), trace a path backward from the \circ site along the tree until a \bullet site is reached, changing all the \circ sites encountered to \bullet sites. Repeat until there are no such \circ sites.

that belong to the backbone, and (3) calculating the mass and radius of gyration of the constructed backbone. A second method of analysis consists in growing large clusters of chemical distance $t \geq 300$. The backbone is then determined for these clusters and the mass and radius of gyration for chemical distances less than and equal to 300 noted.

3. NUMERICAL RESULTS

Our results were derived for clusters grown on the triangular lattice, for which the percolation threshold is known to be exactly $1/2$. We proceed to grow the cluster from a central occupied site, which is considered the initial growth site and labeled t_0 . The neighbors of this initial growth site are occupied with probability p_c and blocked with probability $1 - p_c$. The newly occupied sites are labeled t_1 and are the new growth sites and their *unblocked* vacant neighbors are the potential growth sites at the next step. The growth process is stopped when the cluster has reached a preset chemical radius. We considered *all* clusters grown, provided that there were

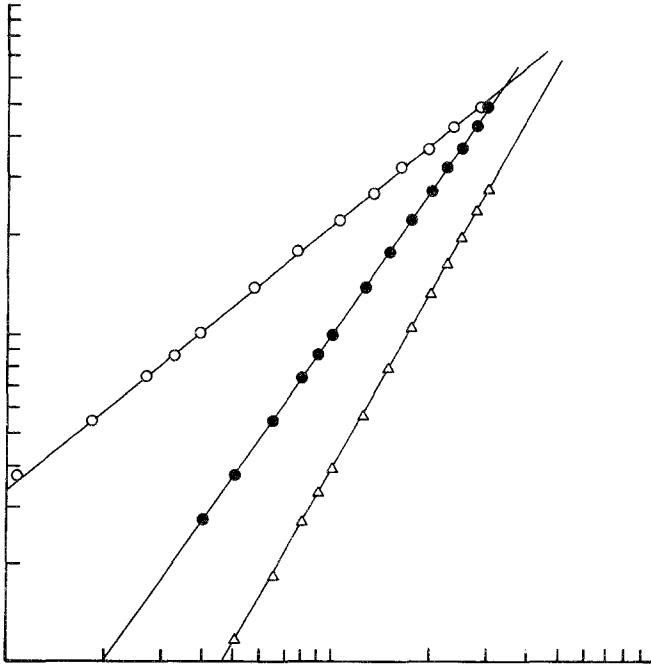


Fig. 2. Log-log plot of (○) mass versus radius of gyration, (●) mass versus chemical distance, (△) and radius of gyration versus chemical distance for the standard backbone.

at least two growth sites at the preset chemical radius. The growth sites that were furthest apart were selected as the two endpoints P_s and P_1 and the recursive algorithm described above was used to determine the backbone of the cluster. We were also able to define a more symmetric backbone, where every growth site at the preset chemical distance was considered as a high potential or endpoint and the central site a low potential on earth. The set of conducting bonds or sites through which there is a flow of current constitute the backbone, which we refer to as the multiple backbone to distinguish it from the standard definition of the backbone. This structure was also considered by Havlin and co-workers, who referred to it as the "skeleton."

The mass M of the backbone is expected to scale with the radius of gyration R_g as

$$M \sim R_g^{d_f^{BB}} \quad (2)$$

In addition, we define the following exponents:

$$M \sim t^{d_f} \quad (3)$$

$$R_g \sim t^{\tilde{\nu}} \quad (4)$$

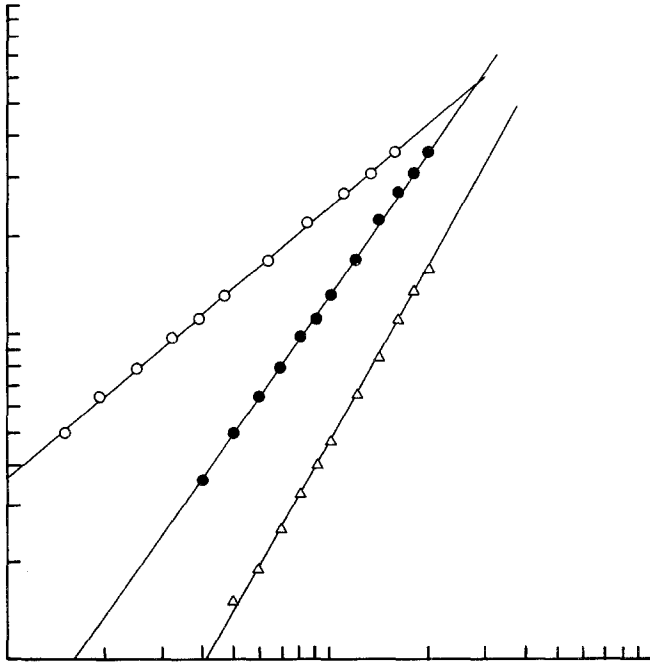


Fig. 3. Log-log plot of (○) mass versus radius of gyration, (●) mass versus chemical distance, and (△) radius of gyration versus chemical distance for the multiple backbone.

From (2)–(4)

$$d_f^{BB} = d_t / \tilde{\nu} \tag{5}$$

where d_t defines the chemical dimension and $\tilde{\nu}$ the relationship between the chemical distance t and the Euclidean distance R_g . Figure 2 shows the results for the standard backbone, which together with a last squares fit of the data lead to $d_f^{BB} = 1.61 \pm 0.02$, $d_t = 1.42 \pm 0.02$, and $\tilde{\nu} = 0.88 \pm 0.01$; thus, $d_{\min} = 1.14 \pm 0.02$. The data for the multiple backbone are displayed in Fig. 3, where a comparable analysis yields $d_f^{BB} = 1.64 \pm 0.05$, $d_t = 1.42 \pm 0.05$, and $\tilde{\nu} = 0.87 \pm 0.04$. We note that our algorithm has allowed us to analyze more realizations, at least 3000 trials for the large systems, and larger systems than had previously been possible ($t \leq 300$) for the backbone. The above computation was done on an HP9000 work station and took ~ 70 CPU hr.

We checked our data for finite-size effects by calculating local slopes from nearby chemical distances, and although this showed large fluctuations, there were no discernible finite-size effects for the chemical distances considered.

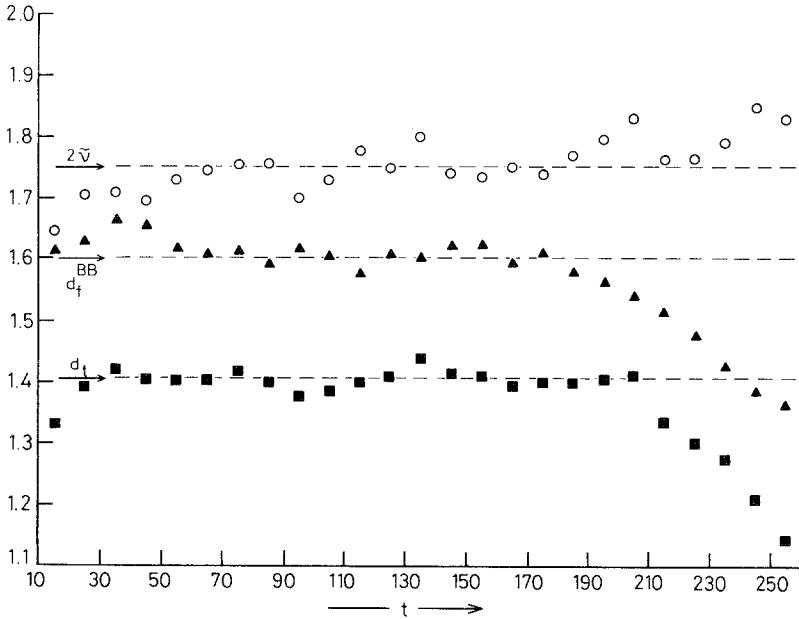


Fig. 4. (■) The chemical dimensionality d_t , (▲) the backbone dimensionality d_f^{BB} , and (○) the chemical distance exponent $\tilde{\nu}$ as a function of chemical distance. These exponents were obtained from the analysis of internal properties of 600 clusters of chemical distance $t = 300$.

The second method of analysis consists in analyzing the properties of only large clusters, i.e., $t \geq 300$. The backbones of these large clusters are determined and the mass and radius of gyration as a function of chemical distance $t \leq 300$ noted. The fractal properties for 600 clusters as a function of chemical distance are shown in Fig. 4. The critical exponents show relatively little fluctuation to $t > 15$ to $t \sim 190$. There is, however, systematic deviation for $t > 200$ due to edge effects. A least squares analysis of the data in the range $t = 20$ to $t = 190$ leads to $d_t = 1.40 \pm 0.01$, $2\tilde{\nu} = 1.75 \pm 0.02$, and $d_f^{\text{BB}} = 1.61 \pm 0.01$. The graph also confirms that there is very little finite-size effect for $t > 30$, although, as noted above, there are strong edge effects.

4. CONCLUSION

We find excellent agreement for the fractal dimensionality of the backbone with the numerical results of Herrmann and Stanley⁽¹⁵⁾ and good agreement with some numerical estimates. The value for d_{min} reported here is in good agreement with the conjecture of Havlin and Nossal,⁽¹⁸⁾ and although our error bars cannot exclude the value reported by Grassberger,⁽¹⁹⁾ we find no evidence for the value $d_{\text{min}} = 1$. Larger systems may place smaller error bars on the various fractal exponents.

Hong *et al.*⁽²⁷⁾ have noted that fractal properties of the backbone show less fluctuation than the equivalent properties on the percolating cluster and therefore the direct measurement of d_{min} is, we believe, as reliable as the numerical results reported by others and perhaps of comparable accuracy. Our numerical results are well represented by the following mnemonics: $d_f^{\text{BB}} = 8/5$, $d_t = 7/5$, and $\tilde{\nu} = 7/8$. DeArcangelis *et al.*⁽⁴⁾ have given a rigorous argument that $d_f^{\text{BB}} \leq d_f^{\text{hull}}$, the fractal dimensionality of the hull of the percolating cluster,⁽²⁵⁾ which is conjectured to be $7/4$.⁽²⁶⁾ It is not without interest to see whether arguments exist to establish exact values for the exponents reported above. We have also constructed the multiple backbone, which appears to be in the same universality class as the standard backbone. The algorithm will be used to determine the fractal properties of the backbone in higher dimensions.

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